Dynamical scaling functions in conserved vector order-parameter systems without topological defects

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We study the growth of order following a zero temperature quench in the one-dimensional XY ($n=2$) and Heisenberg ($n=3$) models and in the two-dimensional $n=4$ model with a conserved order parameter using a Langevin formalism. These systems are characterized by an absence of localized topological defects ($n>d$). Although the structure factor $S(k,t)$ obeys standard dynamical scaling at late times, we show quite convincingly that $S(k,t)$ possesses an exponential tail, violating the generalized Porod's law. We also find that the form of the asymptotic correlation function at small distances exhibits a striking universality.

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There has been much interest of late in the phase ordering of systems with continuous symmetry, following a quench from a high temperature disordered state [1–8]. Most numerical [7] and experimental [8] work has concentrated on systems possessing topological defects for which the number of components of the order parameter, $n\le d$, the spatial dimension. These localized defects, which get introduced into the system as the start of the dynamical evolution, get ironed out as the spins order. For these systems, the equal-time correlation function $g(r,t)$ and hence its Fourier transform, the structure factor $S(k,t)$, obey the simple scaling ansatz [9] at late times,

$$g(r,t)=g(r/R(t)),\quad S(k,t)=R^d(t)F(kR(t)) , \tag{1}$$

where R is the single characteristic length scale associated with the typical distance between defects. In the scaling regime $R\sim t^{1/z}$, where z depends on the conservation laws governing the dynamics and the dimensions [3,4] n and d. The scaling function $g(r/R(t))$ is universal in the sense that it is independent of the nature of the interactions and initial conditions as long as they are short ranged [10]—the form however depends nontrivially on n and d.

How about systems which do not support stable topological defects, i.e., when $n>d$? Is the kinetics of phase ordering different in such cases? Indeed for the exactly solvable $n=\infty$, conserved O(n) model (in arbitrary d), Coniglio and Zannetti [2] (CZ) showed that the scaling ansatz (1) breaks down because of the existence of two marginally different length scales—$R(t)\sim t^{1/4}$ and $[k_m(t)]^{-1}\sim(t/\ln t)^{1/4}$ where $k_m(t)$ is the position of the maximum of $S(k,t)$. This results in a multiscaling form for $S(k,t)\sim R(t)^{\phi(k/k_m(t))d}$, with $\phi(x)=1-(1-x^2)^2$. CZ suggested that this multiscaling behavior might be generic to the asymptotic dynamics of conserved vector order parameters. Following the suggestion by CZ there have been several numerical studies [11–13] in two and three dimensions—the results in every case are inconsistent with multiscaling and support the simple scaling ansatz Eq. (1). Analytic work by Bray and Humayun (BH) [6] using an approximate formalism, originally developed by Mazenko [14], showed that the multiscaling form was a specific feature of the $n=\infty$ model and that the standard scaling ansatz Eq. (1) was restored at any finite n. All these studies are, however, restricted to values of $n\le d$. Mazenko's formalism implicitly assumes the existence of localized defects and analyzes the temporal behavior of the two-point correlation function in terms of fluctuations about this defect configuration. *This still leaves the question open for systems which do not admit localized defects, $n>d$.*

That the physics of $n>d$ systems is different is indicated in the contribution of topological defect configurations to the tail of $S(k,t)$. An approximate calculation of $S(k,t)$ for a nonconserved order parameter [15] based on a singular perturbation expansion, predicted that the asymptotic $S(k,t)$ has a power-law tail of the form $A(n,d)R(t)^{-n}k^{(-d+n)}$ for $kR(t)\gg 1$. This has been verified by numerical simulations [7,11,12] and experiments [8] on systems with $n\le d$. For $n=1$, this is the famous Porod's law [16], long recognized as arising from defect configurations with sharp domain walls. Recent work [17] has explicitly shown that the $k^{-(d+n)}$ tail and the universal amplitude $A(n,d)$ can be understood as a consequence of the existence of topological defects in the order parameter field. Such localized defects have a core where the order parameter profiles vanish, and only exist for $n\le d$. Though it is known that the existence of such localized topological defects is responsible for the power-law tail, there has not been a systematic study of the dependence of the tail of $S(k,t)$ on n and d when $n>d$ (see, however, Ref. [18] for nonconserved systems).

In this paper we investigate the behavior of the equal-time correlation function of conserved order parameter systems which do not possess localized topological defects ($n>d$). We focus on the conserved dynamics of the one-dimensional XY ($n=2$) and the Heisenberg ($n=3$) models and the two-dimensional $n=4$ model, following an instantaneous quench from a disordered configuration to the zero temperature ordered phase using a Langevin formalism. Our main results are the following. We demonstrate that at late times our data is inconsistent with multiscaling and so vindicate the simple scaling ansatz [Eq. (1)]. However, we show that for systems without localized topological defects, $S(k,t)$ possesses an exponentially decaying tail, violating the generalized Porod's law. We find that the one-dimensional Heisenberg and the two-dimensional $n=4$ models exhibit similar features in their structure factors. On the other hand, the one-dimensional XY model exhibits anomalous features both in the growth exponent and in the form of $S(k,t)$. This gives rise to two possibilities—either (i) the $n=2$, $d=2$ model is unique or (ii) its behavior is generic to systems with $n=d+1$. If the latter is true then this would imply that the dynamical behavior of $n=d+1$ systems differs from the $n>d+1$ systems owing to the presence of extended topological defects in the former. In spite of this difference, there is a universal feature shared by all $n>d$ systems, namely, the form of the asymptotic correlation function as small distances is analytic and identical.

We numerically solve the Langevin equation for the $d=2$, $n=4$ and the $d=1$, $n=3$, and $n=2$ models. A zero temperature quench allows us to drop the noise term (whose correlations are proportional to temperature). The resulting equation for the components of the order parameter $\psi_\alpha(\mathbf{r},t)$ with $\alpha=1,\ldots,n$, can be written as

$$\frac{\partial\psi_\alpha}{\partial t}=-\nabla^2[\nabla^2\psi_\alpha+\psi_\alpha(1-\boldsymbol{\psi}^2)] \; . \tag{2}$$

This conserves the total magnetization (integral of the order parameter) and can be derived from a Ginzburg-Landau free-energy functional,

$$F[\boldsymbol{\psi}]=\tfrac{1}{2}\int d\mathbf{x}\left[-r\boldsymbol{\psi}^2+\frac{u}{2}(\boldsymbol{\psi}^2)^2+c(\nabla\boldsymbol{\psi})^2\right] , \tag{3}$$

after a suitable rescaling of the position, time, and order parameter variables. We use the usual Euler discretization method with a $\Delta t=0.025$ and a regular mesh size $\Delta r=1$ to iterate Eq. (2) up to times $t\le t_{\max}=10\,000$ on a system of size $L=1028\Delta r$ in $d=1$, and $L=256\Delta r$ in $d=2$, with periodic boundary conditions. At these late times, we still do not see any sign of finite size effects. We perform ten runs with different realizations of the initial configuration of the components of the order parameter (distributed uniformly between ±0.1) and average over these to obtain $g(r,t)=\langle\boldsymbol{\psi}(\mathbf{r},t)\cdot\boldsymbol{\psi}(0,t)\rangle$ and its Fourier transform $S(k,t)$. We investigate the behavior of the characteristic length scale $r_0(t)$ obtained from the first zero of the correlation function $g(r,t)$.

To distinguish between standard scaling (Eq. 1) and multiscaling, CZ suggested plotting $S(k,t)$ as a function of t for several but fixed values of $x=k/k_m(t)$ where $k_m(t)$ is the position of the peak of $S(k,t)$. On a log-log scale this plot should show several parallel lines with slope d/z if the standard scaling were to be valid, whereas in the case of multiscaling one would see a spread of slopes $\phi(x)d/z$ (z is the dynamical exponent). We find that for all three models considered here, $\phi(x)$, plotted against x, is clustered [19] around 1 as demanded

TABLE I. Exponents and amplitudes of the fitting forms for the characteristic domain size $R(t)$, the correlation function $g(r,t)$, and the structure factor $S(k,t)$.

n	d	$1/z$	ν	$a_2(n,d)$
4	2	0.27 ± 0.01	1.7 ± 0.1	0.54
3	1	0.26 ± 0.01	1.7 ± 0.1	0.98
2	1	0.17 ± 0.01	2.7 ± 0.3	0.78

by the scaling ansatz Eq. (1). There is no evidence of a nontrivial $\phi(x)$, and therefore of multiscaling. This strongly suggests that the multiscaling phenomenon is a singular feature of the $n=\infty$ model, and that for any finite n, standard scaling holds.

Standard scaling implies that there is a single characteristic "domain" size measured by $r_0(t)$ defined above which scales as $t^{1/z}$ at late times. The values of $1/z$ given in Table I for the $n=3$ and $n=4$ systems are consistent with the renormalization group (RG) prediction [3] of $z=4$. This power-law behavior persists over a large interval of times up to late times ($t_{\max}=10\,000$) with no sign of finite size saturation. We note that these systems enter the scaling regime at earlier times in comparison to scalar order parameters. For the conserved XY model in $d=1$, however, our result for $1/z$ (consistent with $z=6$) is different from the RG prediction of $z=4$, but agrees with the exponent seen in Ref [11]. At present, we do not have a clear understanding of this anomalous slow growth for this system, which seems to persist out to late times ($t_{\max}=10\,000$). We note, in passing, that an anomalous slow growth ($z=4$) has been observed [20] in the nonconserved XY model in $d=1$. This can be traced to the fact that the correlation functions depend on the initial conditions imposed on the spin variables. This introduces another length scale in the problem, namely the correlation length of the spins at the initial time. A similar feature should be responsible for the slow growth observed here.

Let us now take a look at the details of the structure factor $S(k,t)$ at late times. Figure 1 indicates that the large k profile of $S(k,t)$ falls faster than any sensible power of k, a violation of the generalized Porod's law. This is consistent with the claim that the power-law tail

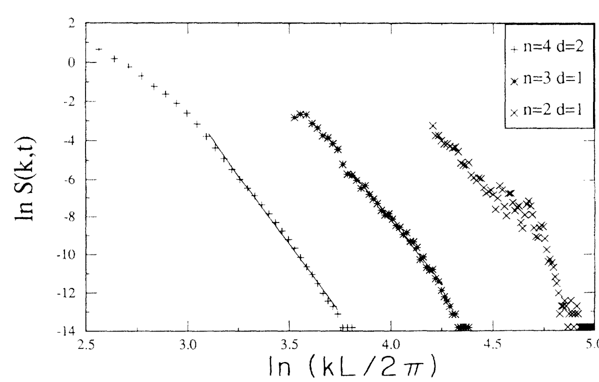

FIG. 1. Violation of Porod's law: a force fit to a power law (solid lines) of the tail of the structure factor $S(k,t)$ gives an exponent ≈ -14 in each case.

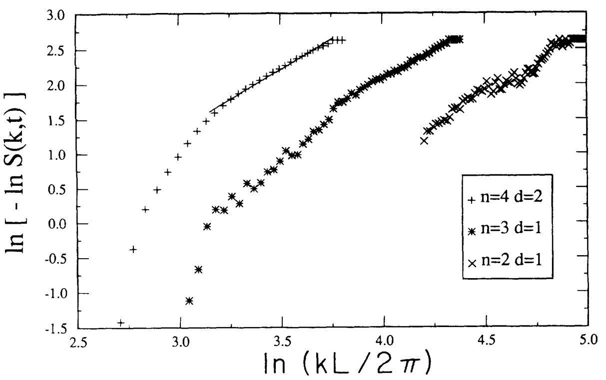

FIG. 2. A fit of the tail of $S(k,t)$ to the form $\exp(-k^{\nu})$ at a late time. The values of ν appear in Table I.

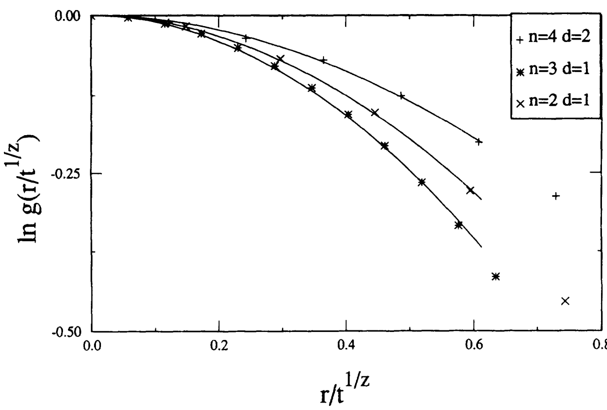

FIG. 3. A fit of $g(x)$ for small x, to the form $\exp[-a_2(n,d)x^2]$ where $x=r/t^{1/z}$. The values of $a_2(n,d)$ are given in Table I.

of $S(k,t)$ observed in $n \le d$ systems is a result of localized topological defects that have been injected into the system at the start of the dynamical evolution. We find that the scaling function $F(kR(r))$ [see Eq. (1)], drops to zero exponentially, $F(x) \sim \exp(-x^{\nu})$ for large x (see Fig. 2). The values of ν are given in Table I. We conjecture that $\nu = \frac{5}{3}$ for all conserved $n > d+1$ models. The $n=\infty$ result [2], $F(x) \sim \exp(-x^4)$ for large x, is clearly singular in n—more evidence that the $n=\infty$ model is pathological. The scaling function $F(x)$ for the $n=2$, $d=1$ system shows a similar exponential decay as before but with a ν consistent with 3 (Table I). The small k behavior of $S(k,t)$ also exhibits interesting features. For the conserved scalar order parameter, the k^4 dependence of $S(k,t)$ for small k follows from essentially dimensional arguments [21] and the assumption that $\mu_k \sim O(1)$, $k \to 0$ (μ_k is the Fourier component of the chemical potential). These arguments should go through unaltered for vector order parameters. Our numerical simulations for the $n=3,4$ systems are indeed consistent with this k^4 behavior. The $n=2$ system is again anomalous—a fit to the $S(k,t)$ data shows a $k^{2.7\pm0.3}$ dependence at small k. It is clear that the behavior of the $n=2, d=1$ model is qualitatively different from the other two models we have studied.

After highlighting the differences between the $n=d+1$ and the $n > d+1$ models, we now show a striking "superuniversal" feature shared by all three models we study. We find that the functional form of the scaling function $g(r/t^{1/z})$ for $r/t^{1/z} \ll 1$ is the same for all three models and conjecture that this should be true for all $n > d$. As shown in Fig. 3, $g(x) \sim \exp[-a_2(n,d)x^2]$ for small x up to $x=0.6$, where the universal coefficients [22] $a_2(n,d)$ appear in Table I. This behavior of the correlation function is distinct from the $n \le d$ systems, where $g(x)$ is nonanalytic at small x, with the leading small-x singularities going as $1-x^n$ for odd n and $1-x^n \ln x$ for even n. It seems reasonable that the extended defects of the $n=2$ model do not make their presence felt at small x (if it does at all). We would expect that the asymptotic form of $g(x)$, for very small x, is insensitive to the conservation of the order parameter. Not surprisingly then, this form is precisely what is obtained in numerical simulations [10,20] of the corresponding nonconserved systems.

In conclusion, we point out that in the absence of localized defects for the $n > d$ systems, the only elementary excitations are spin-wave-like. This is responsible for the exponential correlations seen in these systems. We have provided sufficient numerical evidence to indicate that, unlike for the nonconserved case, the $1/n$ expansion fails completely for conserved systems, since the $n=\infty$ conserved model is singular. One thus needs an exactly solvable nonpathological conserved model about which a systematic perturbation expansion can be performed. A proper theoretical understanding of the numerical results just presented for $n > d$ systems should be an interesting task for the future.

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